An AEGIS-FISST Integrated Detection and Tracking Approach to Space Situational Awareness

Islam I. Hussein

Worcester Polytechnic Institute,
Department of Mechanical Engineering,
Worcester, MA 01609.
Email: ihussein@wpi.edu

Kyle J. DeMars
Carolin Früh

National Academy of Science,
Air Force Research Laboratory,
Space Vehicles Directorate,
3550 Aberdeen Ave SE,
Kirtland AFB, NM 87117.
Email: demars.kyle@ieee.org,
carolin@ece.unm.edu

Richard S. Erwin
Moriba K. Jah

Air Force Research Laboratory,
Space Vehicles Directorate
3550 Aberdeen Ave SE,
Kirtland AFB, NM 87117.

Abstract **Space Situational Awareness (SSA) is composed of three interdependent tasks: discovery of new objects, tracking of detected objects, and characterization of tracked objects. Currently these problems are treated separately and independently of each other, which may result in the non-optimal processing of data, with a corresponding loss of potential information. Given the scarcity of sensors available to perform SSA missions, it is crucial that these resources be used as efficiently as possible. Detection and classification both involve estimation over the space of discrete variables (e.g., existence/nonexistence, satellite mission type), whereas tracking involves estimation over a space of continuously-valued variables (e.g., position and velocity). The current paper uses Finite Set Statistics (FISST) to formulate a hybrid SSA estimation problem, which consists of simultaneously estimating the number of objects and their tracks, in the presence of false alarms, misdetections and sensor noise. The main contribution of the paper is that, in order to reduce the computational burden entailed in FISST, we employ a Gaussian mixture approximation, not to the first-moment (as in GM-PHD) of the full FISST update equations, but apply the approximation directly to the full FISST equations. The specific GM technique we employ is the Adaptive Entropy-based Gaussian-mixture Information Synthesis (AEGIS). The approach is demonstrated on a simplified SSA application example.**

I. Introduction

About five hundred thousand objects one centimeter in size or larger populate the space around earth, which increases the need for new methods and techniques that can contribute to the core elements of Space Situational Awareness (SSA): discovery of new objects, tracking of detected objects, and characterization of tracked objects. Whereas most SSA literature treats detection [1], characterization [2] and tracking [3] independently from each other, the current paper seeks a integrated approach to all these problems. In treating detection, characterization and tracking separately, valuable information can be lost. By taking into account the fact that detection, characterization, and tracking are interdependent, information loss is limited. For example, object type may define some basic shape characteristics that affect motion (e.g., drag coefficient for low Earth orbit spacecraft), and thus provides information about the object's dynamics and its track, and vice versa. Methods that explicitly account for these interdependencies are therefore at an advantage over methods that assume independence between the problems.

Any such integrated approach to the SSA problem must deal with the hybrid (discrete and continuous variables) inference problem that results. The problems of detection and characterization involve estimation over the space of discrete variables (e.g., existence/non-existence, satellite mission type), whereas tracking involves estimation over a space of continuously-valued variables (e.g., position and velocity). Given the potential coupling between the dynamics of these variables as discussed above, the SSA problem is inherently a hybrid estimation problem, where the discrete and continuous variables are dynamically coupled.

Finally, it is important to note that sensor observation time is expensive, and only a few stations worldwide can acquire SSA measurements. In particular, sensor resources at our disposal are scarce in comparison to the size of the SSA tasks at hand (the large number of objects to be catalogued and the large search space). These same limited sensors can be operated in different sensing modes which optimize their detection (for search) or tracking performance characteristics. However, wide-angle measurements (ideal for detection), for example, may result in reduced position accuracy and/or loss of less reflective (with respect to sensor detection wavelength) objects. Decisions on whether to operate the sensor in a track versus a detect mode require an assessment of how much information we expect to gain. This expected information gain will have two (interdependent) components: one that is continuous in nature and the other that is discrete in nature. Therefore, the expected information gain associated with each sensor assignment is of a hybrid nature. The final requirement, therefore, for an effective approach to solving the SSA problem must provide rigorous mechanism for quantifying hybrid information gain for optimal sensor allocation.

The hybrid estimation method we propose for solving

Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE **JUL 2012**	2. REPORT TYPE	3. DATES COVERED **00-00-2012 to 00-00-2012**

4. TITLE AND SUBTITLE **An AEGIS-FISST Integrated Detection and Tracking Approach to Space Situational Awareness**	5a. CONTRACT NUMBER
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Air Force Research Laboratory,Space Vehicles Directorate,3550 Aberdeen Ave SE,Kirtland AFB,NM,87117**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES

Presented at the 15th International Conference on Information Fusion held in Sinapore on 9-12 July 2012. Sponsored in part by Office of Naval Research and Office of Naval Research Global.

14. ABSTRACT

Space Situational Awareness (SSA) is composed of three interdependent tasks: discovery of new objects, tracking of detected objects, and characterization of tracked objects. Currently these problems are treated separately and independently of each other, which may result in the non-optimal processing of data, with a corresponding loss of potential information. Given the scarcity of sensors available to perform SSA missions, it is crucial that these resources be used as efficiently as possible. Detection and classification both involve estimation over the space of discrete variables (e.g., existence/nonexistence, satellite mission type), whereas tracking involves estimation over a space of continuously-valued variables (e.g., position and velocity). The current paper uses Finite Set Statistics (FISST) to formulate a hybrid SSA estimation problem, which consists of simultaneously estimating the number of objects and their tracks, in the presence of false alarms, misdetections and sensor noise. The main contribution of the paper is that, in order to reduce the computational burden entailed in FISST, we employ a Gaussian mixture approximation, not to the first-moment (as in GM-PHD) of the full FISST update equations, but apply the approximation directly to the full FISST equations. The specific GM technique we employ is the Adaptive Entropy-based Gaussian-mixture Information Synthesis (AEGIS). The approach is demonstrated on a simplified SSA application example.

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a REPORT **unclassified**	b ABSTRACT **unclassified**	c THIS PAGE **unclassified**	**Same as Report (SAR)**	**8**	

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

the SSA problem is Finite Set Statistics (FISST) [4], [5]. FISST is a Bayesian approach that allows estimation of hybrid variables, without separating tracking, detection and characterization. FISST naturally suggests a way to generalize existing measures of information gain for purely discrete and purely continuous problems to rigorous hybrid measures of information gain [5], [6], which is important for providing information-theoretic criteria for sensor allocation.

The greatest challenge in implementing FISST in real-time, which is critical to any viable SSA solution, is computational burden. The first-order Probability Hypothesis Density (PHD) approach has been proposed as a computationally tractable approach to applying FISST [5]. PHD can further employ a Gaussian Mixture (GM) or a particle filter approximation to further reduce the computational burden (by removing the need to do a rigorous discretization of the state space) at the expense of simplifying assumptions. In this paper, we develop a GM approximation and apply it, not to the first-order PHD approximation, but to the original full hybrid propagation and update equations derived from FISST. This eliminates any information loss associated with using the first-order PHD approximation. The approach we pursue here is similar in spirit to the concept of the "para-Gaussian" that was very briefly described in [7].

The particular GM technique we use for the approximation is Adaptive Entropy-based Gaussian-mixture Information Synthesis (AEGIS) [8], [9], which is an estimation approach for non-linear continuous dynamical systems. AEGIS implements a GM model representation of the probability density function (pdf) that is adapted online via splitting of the GM components whenever an entropy-based detection of nonlinearity-induced distortions of the Gaussian components is triggered during the forward propagation of the pdf. In doing so, the GM approximation adaptively includes additional components as nonlinearity is encountered and can therefore be used to more accurately approximate the pdf.

To summarize, the main contributions of this paper are (1) the use of the GM AEGIS approach to approximate the complete, un-approximated FISST propagation and update equations, and (2) the application of this approach to the problem of detection and tracking of space objects. The latter is demonstrated via a simple "0-1" object example. The remainder of the paper is organized as follows. We first briefly summarize the theory behind FISST in Section II. Next we briefly summarize AEGIS in Section III. In Section III-A the general AEGIS-FISST approach is summarized. In Section IV, the method is applied to a planar single object SSA detection incorporating the possibility of false alarms, misdetections, and sensor tracking noise. In Section V we present numerical simulation results for the simple SSA problem, demonstrating the capabilities and performance properties of the method. We conclude with future research directions in Section VI.

II. A Brief Introduction to FISST

As opposed to purely discrete or purely continuous Bayesian inference, FISST makes use of set-valued random variables. For example, if we let W be the set of all possible space object types, then $x_d \in W$ is the discrete component of the state that describes a space object's type and $\mathbf{x} \in \mathbb{R}^6$ is the continuous component of the state that specifies the position and velocity of the object, then the set-valued random variable corresponding to the hybrid system state is $X = \{x_d, \mathbf{x}\}$. In a detection and tracking problem, the set-valued hybrid system state $X = (n, \mathbf{X})$, where n is the discrete component that describes the number of objects in the search space and $\mathbf{X} = \left[\mathbf{x}_1^{\mathrm{T}}\ \mathbf{x}_2^{\mathrm{T}}\ \ldots\ \mathbf{x}_n^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{6n}$ describes the positions and velocities of these objects. Notice here the explicit dependence of the dimension of the continuous state space $\mathbb{R}^{6n}$ on the discrete component n of the state. For brevity, we simply write $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ in the detection and tracking problem.

In this paper we only look at the integrated detection and tracking problem. Hence, the state $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ gives the number of objects n and where $\mathbf{x}_i$ describes the position and velocity of object i. We will assume a planar dynamic model. In other words, we have $\mathbf{x}_i \in \mathbb{R}^4$. Similar definitions apply to the set-valued measurement variable $Z = \{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_m\}$ with m sensor returns and where $\mathbf{z}_i \in \mathbb{R}^m$, $i = 1, \ldots, m$, is the value of each return.

Bayes' law takes up exactly the same form in the hybrid FISST approach as it does in purely continuous and purely discrete problems:

$$\begin{aligned} f_{k+1|k}(X|Z^{(k)}) &= \int f_{k+1|k}(X, X') \cdot f_{k|k}(X'|Z^{(k)}) \delta X' \\ f_{k+1|k+1}(X|Z^{(k+1)}) &= \frac{f_{k+1}(Z_{k+1}|X) f_{k+1|k}(X|Z^{(k)})}{f_{k+1}(Z_{k+1}|Z^{(k)})}, \end{aligned} \quad (1)$$

where $Z^{(k+1)} = \{Z_1, \ldots, Z_{k+1}\}$ is the time sequence of measurement sets up to time $k+1$, and where $f_{k+1|k}(X|X')$ is the multi-target Markov transitional density. The function $f_{k+1}(Z|X)$ is the multi-target likelihood function that describes the likelihood of getting a measurement Z_{k+1} given the state X_{k+1}. The first of Eq. (1) is the *prediction step* and the second is the *update or corrector step*. The normalization factor in the update step is given by

$$f_{k+1}(Z_{k+1}|Z^{(k)}) = \int f_{k+1}(Z_{k+1}|X) f_{k+1|k}(X|Z^{(k)}) \delta X. \quad (2)$$

Notice that the integrals are *set integrals*, denoted by the variable of integration δX vice $\mathrm{d}X$ for a standard integral. For multi-target detection and tracking, the set integral of a scalar-valued set function $g(X)$ is defined to be the integral of g over the continuous component, summed over all possible discrete values [4], [5]:

$$\begin{aligned} \int g(X) \delta X &= g(X = \emptyset) \\ &\quad + \sum_{i=1}^{\infty} \frac{1}{i!} \int g(\{\mathbf{x}_1, \ldots, \mathbf{x}_i\}) \mathrm{d}\mathbf{x}_1 \cdots \mathrm{d}\mathbf{x}_i, \end{aligned} \quad (3)$$

where the factorial coefficient takes into account all the different possible orderings of X as evaluated in the function g. The need for this factor accounts for the data-association problem [5].

III. A Brief Introduction to AEGIS

Most implementations of Gaussian mixture propagation algorithms assume that the component weights remaining constant over the duration of the propagation span, and so do not incorporate a method for online refinement of the Gaussian mixture. Methods, such as the Adaptive Gaussian Mixture provide a method for adaptive weight variation without a mechanism to vary (coarsen or refine) the number of Gaussian components [10]. The AEGIS filter approaches the problem of adapting the weights of the GM pdf by monitoring nonlinearity-induced distortions of the Gaussian components during the propagation of the pdf and using a splitting algorithm to increase the accuracy of linearization, thereby allowing the filter to modify the GM components (weight value and cardinality) in such a way so as to avoid significant linearization errors.

1) Dynamical System Model: Many systems of interest fall under the broad classification of nonlinear systems. An estimation algorithm which exploits at least some characteristics of the nonlinearities is preferable to approximating the problem as that of a linear one. Consider the nonlinear dynamical system governed by the differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), t)\,, \quad \mathbf{x}(t_0) = \mathbf{x}_0\,, \tag{4}$$

where $\mathbf{x}(t)$ is the non-set valued, continuous state of the system, $\mathbf{f}(\cdot)$ represents the sufficiently differentiable nonlinear dynamics of the system, and $\mathbf{x}_0$ is the initial condition. The initial condition is assumed to be random with pdf $f_{0|0}(\mathbf{x})$.

2) Detecting Nonlinearity Induced Distortions to the GM during Propagation: An integral aspect of the AEGIS propagation scheme is the detection of nonlinearity during the propagation of uncertainty. The method employed in the AEGIS approach is based on a property derived from the differential entropy for linearized dynamical systems that allows for the detection of nonlinearity without employing linearization-based methods. It can be shown that the differential entropy for a linearized, Gaussian system evolves as [11], [12]

$$\dot{\mathrm{H}}\left(\mathbf{x}\right) = \mathrm{trace}\left\{\mathbf{F}(\hat{\mathbf{x}}(t), t)\right\}\,, \tag{5}$$

where $\mathbf{F}(\hat{\mathbf{x}}(t), t)$ is the dynamics Jacobian matrix. The value of the entropy for a linearized system can be determined by numerically integrating Eq. (5) with an appropriate initial condition and requiring only the evaluation of the trace of the linearized dynamics Jacobian. In parallel, a nonlinear implementation of the integration of the covariance matrix (e.g. the Unscented Kalman Filter (UKF) [13]) is considered, which allows a nonlinear determination of the differential entropy. Any deviation between these values of entropy therefore serves as an indication that nonlinearity is impacting the solution. When the difference between the linearized-predicted entropy and the nonlinear computation of the entropy exceeds a user-defined threshold, nonlinearity has been detected in the propagation of the uncertainty, the propagation is halted, a splitting algorithm is applied to the Gaussian distribution, and propagation resumes with the adapted distribution.

3) Propagation: Consider the time-propagation of the pdf and consider the time interval $t \in [t_k, t_{k+1}]$. It is desired to approximate the conditional pdf at time t_k via

$$f_{k|k}(\mathbf{x}|Z^k) = \sum_{i=1}^{\bar{N}} \bar{w}_i N_i(\mathbf{x}; \bar{\boldsymbol{\mu}}_i, \bar{\mathbf{P}}_i), \tag{6}$$

where $N_i(\mathbf{x}; \bar{\boldsymbol{\mu}}_i, \bar{\mathbf{P}}_i)$ is a Gaussian distribution in $\mathbf{x}$ with mean $\bar{\boldsymbol{\mu}}_i$ and covariance $\bar{\mathbf{P}}_i$, and where $\bar{w}_i$ is the weight of the i^{th} GM component with $\sum_{i=1}^{\bar{N}} \bar{w}_i = 1$. The propagated pdf is then given by

$$f_{k+1|k}(\mathbf{x}|Z^k) = \sum_{i=1}^{\tilde{N}} \tilde{w}_i N_i(\mathbf{x}; \tilde{\boldsymbol{\mu}}_i, \tilde{\mathbf{P}}_i), \tag{7}$$

where $\tilde{w}_i$, $\tilde{\boldsymbol{\mu}}_i$ and $\tilde{\mathbf{P}}_i$, $i = 1, \ldots, \tilde{N}$, are the weight, mean and covariance of the of the i^{th} propagated GM component. It should be noted that due to the splitting algorithm described above, the number of components in $f_{k+1|k}(\mathbf{x}|Z^k)$, given by $\tilde{N}$, may now be different than the number of components in $f_{k|k}(\mathbf{x}|Z^k)$, given by $\bar{N}$.

To propagate the pdf forward in time, a nonlinear algorithm such as the UKF is applied to each component of the GM pdf. Simultaneously, the linearized differential entropy is propagated as described previously. Each component is monitored for deviations in the nonlinear and linear predictions of the entropy. Once nonlinearity is detected on a component, the propagation is halted at time t_s, where $t_k \leq t_s \leq t_{k+1}$. If $t_s \neq t_{k+1}$, then a splitting step is performed on the component for which nonlinearity was detected.[1] That is, if nonlinearity was detected in the j^{th} component at time t_s, then the j^{th} component is replaced at time t_s by

$$\bar{w}_j N_j(\mathbf{x}; \bar{\boldsymbol{\mu}}_j, \bar{\mathbf{P}}_j) \approx \sum_{r=1}^{G} w_r N_r(\mathbf{x}; \boldsymbol{\mu}_r, \mathbf{P}_r) \tag{8}$$

where t_s has been omitted for brevity, and the replacement component weights, means, and covariances are computed using the Gaussian splitting algorithm discussed in [11]. After the splitting step has been performed, return to Eq. (7) with $t_k = t_s$ and $\bar{N} \leftarrow \bar{N} + G - 1$ components in the Gaussian mixture, then continue until $t_s = t_{k+1}$ is reached. Once $t_s = t_{k+1}$, the propagation step has been completed with $\tilde{N}$ components having weights $\tilde{w}_i$, means $\tilde{\boldsymbol{\mu}}_i$, and covariances $\tilde{\mathbf{P}}_i$, which allows the *a priori* GM pdf to be evaluated via Eq. (6).

A. AEGIS based Approximation of FISST: AEGIS FISST

The AEGIS-FISST approach relies on the observation that if the (conventional non-hybrid) prior $f_{k|k}(\mathbf{x}|Z^{(k)})$ is a GM, then the posterior multi-target pdf will also be a sum of Gaussian mixtures from which one can extract the (conventional) posterior pdf $f_{k+1|k+1}(\mathbf{x}|Z^{(k+1)})$ (also a GM) under the hypothesis that an object exists. This observation is what

[1] Without loss of generality, nonlinearity is assumed to be detected on only one component. If more components detect nonlinearity, the same process is applied to each component individually.

distinguishes the proposed method from a GM implementation of the PHD approach [5]. PHD relies on approximating the multi-target posterior pdf by its first-order moment PHD. After that, a GM is used to approximate the PHD. In our approach, we circumvent the first step of approximating the multi-target pdf by its PHD and realize that the Gaussian mixture property is preserved in the update step. Thus, one need not approximate the multi-target pdf by its first-order PHD and the only approximation one performs is that of using a GM to model the prior pdf.

IV. A Single-Object Illustrative Example

A. The Basic FISST Propagation and Update Equations

For the sake of brevity and ease of exposition, we consider a problem in which there can exist at most one object in the search space. The sensors are assumed imperfect with a non-zero probability of false alarm due to clutter (small objects that are of no interest, thruster exhaust plumes, environmentally-induced sensor responses, etc), and with a non-unity probability of detection when the object is within a pre-specified field-of-view, and a zero probability of detection when the object is outside this field-of-view. In tracking an object, the sensor adds some tracking noise. Without loss of generality, the sensor is assumed to measure the position and velocity of an object directly.

For the single object case with at most a single source of clutter, one can use a Bernoulli distribution [14] for the various multi-target densities as we will show below. Extension to the arbitrary number of objects can be addressed using a Poisson distribution model for the number of objects as well as clutter sources.

Under these assumptions and given measurements $Z^(k)$ up to time step k, the *multi target prior density function* has the form

$$f_{k|k}(X|Z^{(k)}) \quad = \quad \begin{cases} 1-p_k & \text{if } X=\emptyset \\ p_k \cdot f_{k|k}(\mathbf{x}|Z^{(k)}) & \text{if } X=\{\mathbf{x}\} \\ 0 & \text{if } |X|\geq 2 \end{cases} \quad (9)$$

where p_k is the prior probability that the object exists at time k (we will henceforth drop the subscript k) with prior conventional pdf given by $f_{k|k}(\mathbf{x}|Z^{(k)})$.

Remark. In composing a multi-target pdf, say for the the case that $X = \{\mathbf{x}\}$ (i.e., the hypothesis that an object exists in the search space with a state $\mathbf{x}$), one considers $f_{k|k}(X = \{\mathbf{x}\}|Z^{(k)})$ as the "probability" of an object existing with true probability p *and* having the object's continuous state having (conventional) pdf $f_{k|k}(\mathbf{x}|Z^{(k)})$. Hence, $f_{k|k}(X|Z^{(k)}) = p \cdot f_{k|k}(\mathbf{x}|Z^{(k)})$ as shown in the second line in Eq. (9). The "and" operation was translated into a product of the discrete probability p and the continuous pdf $f_{k|k}(\mathbf{x}|Z^{(k)})$. Likewise, the probability that $X = \emptyset$ is thus simply $1-p$. One can check that $\int f_{k|k}(X|Z^{(k)})\delta X = 1$ and, hence, $f_{k|k}(X|Z^{(k)})$ is a valid multi-target pdf.

Since we make the simplifying assumption that at most a single object exists in the search space, the *multi target transitional Markov density function* is given by

$$f_{k+1|k}(X|\emptyset) = \begin{cases} 1 & \text{if } X=\emptyset \\ 0 & \text{if } |X|\geq 1 \end{cases}$$

$$f_{k+1|k}(X|\{\mathbf{x}'\}) = \begin{cases} 1-p_s & \text{if } X=\emptyset \\ p_s \cdot f_{k+1|k}(\mathbf{x}|\mathbf{x}') & \text{if } X=\{\mathbf{x}\} \\ 0 & \text{if } |X|\geq 2 \end{cases} \quad (10)$$

where p_s is the probability that the object servives in the search space from time step k to time step $k+1$ and where $f_{k+1|k}(\mathbf{x}|\mathbf{x}')$ is the conventional Markov transitional density function that describes the likelihood of transitioning to the state $\mathbf{x}$ from the state $\mathbf{x}'$ under the hypothesis that an object with state $\mathbf{x}'$ exists. One can add a birth model, but we omit this here for the sake of transparency of exposition.

Applying the basic definition of a set integral, for the *predicted multi target density function*, we have

$$\begin{aligned} & f_{k+1|k}(X=\emptyset|Z^{(k)}) \\ &\quad = \int f_{k+1|k}(X=\emptyset|X')f_{k|k}(X'|Z^{(k)})\delta X' \\ &\quad = f_{k+1|k}(\emptyset|\emptyset)\cdot f_{k|k}(\emptyset|Z^{(k)}) \\ &\qquad + \int f_{k+1|k}(\emptyset|\{\mathbf{x}'\}\cdot f_{k|k}(\{\mathbf{x}'\}|Z^{(k)})\mathrm{d}\mathbf{x}' \\ &\quad = (1-p)+p\int(1-p_s)f_{k|k}(\mathbf{x}'|Z^{(k)})\mathrm{d}\mathbf{x}' \\ &\quad = (1-p)+p(1-p_s) \end{aligned} \quad (11)$$

and (we have omitted the calculation for the second expression for the sake of brevity)

$$f_{k+1|k}(X=\{\mathbf{x}\}|Z^{(k)}) = p_s p f_{k+1|k}(\mathbf{x}|Z^{(k)}). \quad (12)$$

One can check that $\int f_{k+1|k}(X)\delta X = 1$.

Let the probability of detection be p_D and the probability of false alarm be p_F. These two probabilities are generally a function of the location of the object (whether it is in the field-of-view or not). Then the *multi target likelihood function* is given by

$$\begin{aligned} f_{k+1}(Z_{k+1}=\emptyset|\emptyset) &= 1-p_F \\ f_{k+1}(Z_{k+1}=\{\mathbf{z}\}|\emptyset) &= p_F g(\mathbf{z}) \\ f_{k+1}(Z_{k+1}=\emptyset|\{\mathbf{x}\}) &= (1-p_F)(1-p_D) \\ f_{k+1}(Z_{k+1}=\{\mathbf{z}\}|\{\mathbf{x}\}) &= p_F(1-p_D)g(\mathbf{z}) \\ &\quad +p_D(1-p_F)f_{k+1}(\mathbf{z}|\mathbf{x}) \\ f_{k+1}(Z_{k+1}=\{\mathbf{z}_1,\mathbf{z}_2\}|\{\mathbf{x}\}) &= p_F p_D\big[g(\mathbf{z}_1)f_{k+1}(\mathbf{z}_2|\mathbf{x}) \\ &\quad +g(\mathbf{z}_2)f_{k+1}(\mathbf{z}_1|\mathbf{x})\big] \end{aligned} \quad (13)$$

where $g(\mathbf{z})$ is the spatial likelihood clutter distribution function that a clutter generates the measurement $\mathbf{z}$.

Remark. In order to understand where the components of the multi-target likelihood function originate from, consider for example the last component $f_{k+1}(Z_{k+1} = \{\mathbf{z}_1, \mathbf{z}_2\}|\{\mathbf{x}\})$. Given that an object exists in the search space, this is the "probability" that two measurements are registered. The only two possibilities are that $\mathbf{z}_1$ was generated by a clutter source *and* $\mathbf{z}_2$ by an object, *or* vice-versa. This gives the sum

in the bracketed term. For that scenario (i.e., two sensor returns) to occur a detection of the object (hence, the p_D coefficient outside the bracketed term) as well as a false alarm triggered by clutter (hence, the p_F coefficient outside the bracketed term) must take place. One can check that $\int f_{k+1}(Z_{k+1}|\emptyset)\delta Z_{k+1} = \int f_{k+1}(Z_{k+1}|\{\mathbf{x}\})\delta Z_{k+1} = 1$.

The *multi target Bayes factor* in the denominator of the corrector step (the second of Eq. (1) is given by

$$\begin{aligned} & f_{k+1}(Z_{k+1} = \emptyset|Z^{(k)}) \\ & = \int f_{k+1}(Z = \emptyset|X) f_{k+1|k}(X|Z^{(k)})\delta X \\ & = (1-p_F)\left((1-p) + p(1-p_s)\right) + (1-p_F)(1-p_D)p_s p, \end{aligned} \tag{14}$$

$$\begin{aligned} & f_{k+1}(Z_{k+1} = \{\mathbf{z}\}|Z^{(k)}) = \int f_{k+1}(\{\mathbf{z}\}|X) f_{k+1|k}(X|Z^{(k)})\delta X \\ & = \left[p_F\left((1-p) + p(1-p_s)\right) + p_s p p_F (1-p_D)\right] g(\mathbf{z}) \\ & \quad + p_s p p_D (1-p_F) f_{k+1}(\mathbf{z}), \end{aligned} \tag{15}$$

and

$$\begin{aligned} & f_{k+1}(Z_{k+1} = \{\mathbf{z}_1, \mathbf{z}_2\}|Z^{(k)}) \\ & = \int f_{k+1}(Z = \{\mathbf{z}_1, \mathbf{z}_2\}|X) f_{k+1|k}(X|Z^{(k)})\delta X \\ & = p_s p p_F p_D \left(g(\mathbf{z}_1) f(\mathbf{z}_2) + g(\mathbf{z}_2) f(\mathbf{z}_1)\right), \end{aligned} \tag{16}$$

where $f_{k+1}(\mathbf{z})$ is the conventional Bayes factor given that an object exists. Again, one can check that $\int f_{k+1}(Z_{k+1}|Z^{(k)})\delta Z_{k+1} = 1$.

Putting all the above together for the *multi target corrector step*, we get the following.

If $Z_{k+1} = \emptyset$ (i.e., no sensor return at time step $k+1$) the posterior multi-target pdf component corresponding to the hypothesis that no object exists is given by:

$$\begin{aligned} & f_{k+1|k+1}(X = \emptyset|Z_{k+1} = \emptyset) \\ & \quad = \frac{f_{k+1}(Z_{k+1} = \emptyset|X_{k+1} = \emptyset) f_{k+1|k}(X = \emptyset|Z^{(k)})}{f_{k+1}(Z_{k+1} = \emptyset)} \\ & \quad = \frac{(1-p_F)\left((1-p) + p(1-p_s)\right)}{f_{k+1}(Z_{k+1} = \emptyset)}. \end{aligned} \tag{17}$$

Likewise for the case that an object exists with state $\mathbf{x}$, we obtain

$$\begin{aligned} & f_{k+1|k+1}(X = \{\mathbf{x}\}|Z_{k+1} = \emptyset) \\ & \quad = \frac{(1-p_D)(1-p_F)p_s p f_{k+1|k}(\mathbf{x}|Z^{(k)})}{f_{k+1}(Z_{k+1} = \emptyset)}. \end{aligned} \tag{18}$$

Omitting the detailed derivation, if $Z_{k+1} = \{\mathbf{z}\}$ we have:

$$\begin{aligned} & f_{k+1|k+1}(X = \emptyset|Z_{k+1} = \{\mathbf{z}\}) \\ & \quad = \frac{p_F\left((1-p) + p(1-p_s)\right) g(\mathbf{z})}{f_{k+1}(Z_{k+1} = \{\mathbf{z}\}))} \end{aligned} \tag{19}$$

and

$$\begin{aligned} & f_{k+1|k+1}(X = \{\mathbf{x}\}|Z_{k+1} = \{\mathbf{z}\}) = \\ & \frac{\left(p_D(1-p_F) f_{k+1}(\mathbf{z}|\mathbf{x}) + p_F(1-p_D) g(\mathbf{z})\right) p_s p f_{k+1|k}(\mathbf{x}|Z^{(k)})}{f_{k+1}(Z_{k+1} = \{\mathbf{z}\})}. \end{aligned} \tag{20}$$

If $Z_{k+1} = \{\mathbf{z}_1, \mathbf{z}_2\}$ we have:

$$f_{k+1|k+1}(X = \emptyset|Z_{k+1} = \{\mathbf{z}_1, \mathbf{z}_2\}) = 0. \tag{21}$$

Remark. Equation (21) makes perfect sense since getting two returns (under the assumption that at most a single object exists) ensures that at least one of them was generated by an existing object (and, hence, that the probability that no object exists is zero), and

$$\begin{aligned} & f_{k+1|k+1}(X = \{\mathbf{x}\}|Z_{k+1} = \{\mathbf{z}_1, \mathbf{z}_2\}) = \\ & \frac{p_F p_D (g(\mathbf{z}_1) f_{k+1}(\mathbf{z}_2|\mathbf{x}) + g(\mathbf{z}_2) f_{k+1}(\mathbf{z}_1|\mathbf{x})) p_s p f_{k+1|k}(\mathbf{x}|Z^{(k)})}{f_{k+1}(Z_{k+1} = \{\mathbf{z}_1, \mathbf{z}_2\})}. \end{aligned} \tag{22}$$

This second component will in fact integrate to one. The integral of $f_{k+1|k+1}(X = \{\mathbf{x}\}|Z_{k+1} = \{\mathbf{z}_1, \mathbf{z}_2\})$ gives the posterior discrete probability that an object exists. Hence, that quantity will have to integrate to 1.

B. The AEGIS FISST Approximation for the Single Object Problem

We will assume standard (unperturbed) Keplerian object dynamics which can be modeled in the form (4). Substituting (6) and (7) into Eq. (17)-(22) to give an updated set of posterior Gaussian sum parameters at time $k+1$. The posterior Gaussian sum has the form

$$\begin{aligned} f_{k+1|k+1}(\mathbf{x}|Z^{k+1}) &= \sum_{i=1}^{\hat{N}} \hat{w}_i N_i(\mathbf{x}; \hat{\boldsymbol{\mu}}_i, \hat{\mathbf{P}}_i) \\ &= \frac{f_{k+1|k+1}(X = \{\mathbf{x}\}|Z_{k+1})}{p(t_{k+1})}, \end{aligned} \tag{23}$$

with $\sum_{i=1}^{\hat{N}} \hat{w}_i = 1$ and where $p(t_{k+1})$ is the discrete posterior probability that an object exists and is given by [5]

$$p(t_{k+1}) = \int f_{k+1|k+1}(X = \{\mathbf{x}\}|Z_{k+1})\mathrm{d}\mathbf{x}. \tag{24}$$

Equation (23) can be used to extract the components of the posterior GM under the hypothesis that an object exists.

If $Z_{k+1} = \emptyset$, since there is no return, the "updated" components are simply the same as those of the propagated GMs. This can readily be verified by substituting $Z_{k+1} = \emptyset$ into the update equations from the previous section. Hence, the updated GMs are given by:

$$\begin{aligned} \hat{N} &= \tilde{N}, \\ \hat{w}_i &= \tilde{w}_i, \\ \hat{\boldsymbol{\mu}}_i &= \tilde{\boldsymbol{\mu}}_i \\ \hat{\mathbf{P}}_i &= \tilde{\mathbf{P}}_i, \end{aligned} \tag{25}$$

$i = 1, \ldots, \tilde{N}$.

If $Z_{k+1} = \{\mathbf{z}\}$ (i.e., the sensor gets one return), the resulting posterior GM can be discretized down into two separate sums: (1) an update of the posterior under the hypothesis that the measurement came from an existing object, and (2) an update under the hypothesis that the measurement came from a clutter and, in which case, the propagated GM is left unchanged for this case. Total number of components would then be $\hat{N} = 2\tilde{N}$. Substituting $Z_{k+1} = \{\mathbf{z}\}$ and Eq. (23) into the update equations for the single-object problem, one can check that the posterior components of the GM are given by (we split the components into two groups for the two cases pointed out earlier in the paragraph):

$$\begin{aligned}
\hat{N} &= 2\tilde{N} \\
\hat{w}_i^1 &= \frac{p_D(1-p_F)p_s p N_i\left(\mathbf{z}; \mathbf{H}\tilde{\boldsymbol{\mu}}_i, \mathbf{R} + \mathbf{H}\tilde{\mathbf{P}}_i\mathbf{H}^{\mathrm{T}}\right)\tilde{w}_i}{d_1(\mathbf{z})} \\
\hat{w}_i^2 &= \frac{p_F(1-p_D)p_s p g(\mathbf{z})\tilde{w}_i}{d_1(\mathbf{z})}, \\
\hat{\boldsymbol{\mu}}_i^1 &= (\tilde{\mathbf{P}}_i^{-1} + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H})^{-1}(\tilde{\mathbf{P}}_i^{-1}\tilde{\boldsymbol{\mu}}_i + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{z}) \\
\hat{\boldsymbol{\mu}}_i^2 &= \tilde{\boldsymbol{\mu}}_i \\
\hat{\mathbf{P}}_i^1 &= \left(\tilde{\mathbf{P}}_i^{-1} + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H}\right)^{-1} \\
\hat{\mathbf{P}}_i^2 &= \tilde{\mathbf{P}}_i,
\end{aligned} \tag{26}$$

$i = 1, \ldots, \tilde{N}$, where

$$d_1(\mathbf{z}) = p_D(1-p_F)p_s p f_{k+1}(\mathbf{z}) + p_F(1-p_D)p_s p g(\mathbf{z}),$$

and where

$$f_{k+1}(\mathbf{z}) = \sum_{i=1}^{\tilde{N}} \tilde{w}_i N_i(\mathbf{z}; \mathbf{H}\tilde{\boldsymbol{\mu}}_i, \mathbf{R} + \mathbf{H}\tilde{\mathbf{P}}_i\mathbf{H}^{\mathrm{T}}).$$

In the above equations $\mathbf{H}$ is the linearization of $h(\mathbf{x})$ evaluated at the propagated value of the state estimate $\tilde{\mathbf{x}}_i$.

If $Z_{k+1} = \{\mathbf{z}_1, \mathbf{z}_2\}$ (i.e., the sensor receives two returns), the resulting posterior Gaussian sum can be broken down into two separate sums: (1) an update of the posterior under the hypothesis that the first measurement came from a clutter and the second from an existing object, and (2) an update of the posterior under the hypothesis that the second measurement came from a clutter and the first from an existing object. The total number of components in the posterior GM would then be $\hat{N} = 2\tilde{N}$ with components

$$\begin{aligned}
\hat{N} &= 2\tilde{N} \\
\hat{w}_i^1 &= \frac{p_F p_D p_s p g(\mathbf{z}_1) N_i\left(\mathbf{z}_2; \mathbf{H}\tilde{\boldsymbol{\mu}}_i, \mathbf{R} + \mathbf{H}\tilde{\mathbf{P}}_i\mathbf{H}^{\mathrm{T}}\right)\tilde{w}_i}{d_2(\mathbf{z}_1, \mathbf{z}_2)} \\
\hat{w}_i^2 &= \frac{p_F p_D p_s p g(\mathbf{z}_2) N_i\left(\mathbf{z}_1; \mathbf{H}\tilde{\boldsymbol{\mu}}_i, \mathbf{R} + \mathbf{H}\tilde{\mathbf{P}}_i\mathbf{H}^{\mathrm{T}}\right)\tilde{w}_i}{d_2(\mathbf{z}_1, \mathbf{z}_2)} \\
\hat{\boldsymbol{\mu}}_i^1 &= (\tilde{\mathbf{P}}_i^{-1} + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H})^{-1}(\tilde{\mathbf{P}}_i^{-1}\tilde{\boldsymbol{\mu}}_i + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{z}_2) \\
\hat{\boldsymbol{\mu}}_i^2 &= (\tilde{\mathbf{P}}_i^{-1} + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H})^{-1}(\tilde{\mathbf{P}}_i^{-1}\tilde{\boldsymbol{\mu}}_i + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{z}_1) \\
\hat{\mathbf{P}}_i^1 &= \hat{\mathbf{P}}_i^2 = \left(\tilde{\mathbf{P}}_i^{-1} + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H}\right)^{-1},
\end{aligned} \tag{27}$$

where

$$d_2(\mathbf{z}_1, \mathbf{z}_2) = p_F p_D \left(g(\mathbf{z}_1)p_s p f_{k+1}(\mathbf{z}_2) + g(\mathbf{z}_2)p_s p f_{k+1}(\mathbf{z}_1)\right),$$

and where $f_{k+1}(\mathbf{z}_j),\ j = 1, 2$ for the Gaussian sum approximation can be shown to be

$$f_{k+1}(\mathbf{z}_j) = \sum_{i=1}^{\tilde{N}} \tilde{\mathbf{w}}_i N_i(\mathbf{z}_j; \mathbf{H}\tilde{\boldsymbol{\mu}}_i, \mathbf{R} + \mathbf{H}\tilde{\mathbf{P}}_i\mathbf{H}^{\mathrm{T}}). \tag{28}$$

The above posterior Gaussian sum parameters become the prior at time step $k+1$. The above GM parameters can be used to compute the posterior multi-target pdf $f_{k+1|k+1}(X|Z_{k+1})$. One can then employ the Marginal Multitarget (MaM) estimator or the Joint Multitarget estimator to solve for the state estimate $\hat{\mathbf{x}}_{k+1}$ if an object is determined to be existing [5]. For this simple single-object case, both methods are equivalent and the state estimate $\hat{\mathbf{x}}_{k+1}$ is simply the the maximum of $f_{k+1|k+1}(X|Z_{k+1})$.

In general detection and estimation problems, the first step in the MaM estimator is to compute the maximum a posteriori (MAP) estimate of the number of objects, which is given by:

$$\hat{n} = \mathrm{round}(\arg\sup_{n} f_{k|k}(n|Z^{(k)})), \tag{29}$$

where $f_{k|k}(n|Z^{(k)})$ is the cardinality distribution [5]. Given $\hat{n}$, the second step is to solve for the maximum a posteriori estimate of the state

$$\hat{\mathbf{x}}^{\mathrm{MaM}} = \arg\sup_{\{\mathbf{x}_1,\ldots,\mathbf{x}_{\hat{n}}\}} f_{k|k}(\{\mathbf{x}_1, \ldots, \mathbf{x}_{\hat{n}}|Z^{(k)}\}). \tag{30}$$

The MaM estimator is Bayes-optimal but is not known if it is statistically consistent (i.e., whether it converges to the correct state in the static case) [4].

For the (at most) single object object problem, $\hat{n} = 1$ if the posterior discrete probability $p(t_{k+1})$ is larger than 0.5. If $p(t_{k+1}) > 0.5$, the state estimate corresponds to the maximizer of the posterior multi-target pdf $f_{k+1|k+1}(X = \{\mathbf{x}\}|Z_{k+1})$.

V. Numerical Simulation Results

In this section we demonstrate the single-object approach, outlined in the previous section, on a simple SSA problem. We assume that there can be at most a single object. If the orbiting object exists, it satisfies unperturbed two-body planar Keplerian dynamics [15]. We will assume that if an object indeed exists, its probability of survival is $p_s = 1$.

We set the problem up such that an object initially exists with probability one and set the prior assumed by AEGIS-FISST to be $p_0 = 1$. The object true initial state is such that its apoapsis is 42,100 km and the eccentricity is 0.2 (specifying these determines the initial position and velocity of the object [15]). The prior distribution fed into the AEGIS-FISST algorithm was assumed to have a mean identical to the true object location, but with a position variance of 1 km^2 and a velocity variance of 1 m^2/s^2. Hence, the initial prior is a single-component Gaussian mixture distribution as required by AEGIS-FISST.

The clutter distribution $g(\mathbf{z})$ at time t_k that generates false alarms is assumed to be Gaussian centered around the current location and velocity of a small, insignificant object such that: $g(\mathbf{z}) \sim N(\mathbf{z}; \mathbf{x}_c, \mathbf{R}_c)$, where $\mathbf{x}_c$ is the state of the clutter source and is assumed to satisfy (unperturbed) Keplerian dynamics itself with apoapsis of 42,000 km and an eccentricity of 0.6. The clutter measurement error covariance is given by:

$$\mathbf{R}_c = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 10^{-3} & 0 \\ 0 & 0 & 0 & 10^{-3} \end{bmatrix}.$$

Modeling clutter-generation as the result of a single clutter object source in orbit around Earth is an assumption that we introduce for the sake of simplicity. A more realistic Poisson model generating multiple clutter points can be introduced for increased modeling fidelity at the cost of increased simulation numerical burden.

The sensor is assumed to be located on the surface of the Earth, and is hence rotating with respect to the inertial frame. The sensor field-of-view is assumed to be ± 20 degrees from the local vertical. If the object is within the field-of-view, its probability of detection $p_D = 0.6$, and zero otherwise. If the clutter source is within the field-of-view, the probability of false alarm is $p_F = 0.3$, and zero otherwise. The sensor position tracking error is assumed to be zero-mean with covariance 0.500 km^2 and we assume that the sensor measures the cartesian components of velocity corrupted by zero-mean noise with a variance of $1 \text{ m}^2/\text{s}^2$. Hence, the output matrix $\mathbf{H}$ is simply the identity matrix and

$$\mathbf{R} = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 10^{-3} & 0 \\ 0 & 0 & 0 & 10^{-3} \end{bmatrix}.$$

At the end of each time step, using the posterior GM components, we compute the posterior multi-target pdf $f_{k+1|k+1}(X|Z_{k+1})$ and use Matlab's unconstrained maximization function `fminunc` to find the MaM state estimate.

The simulation was run for twelve orbital periods (approximately 9 days) with a sensor sampling rate of a single measurement every 200 seconds. Figure 1 shows the relative position error between the estimated and true object position. In the figure, a red star indicates the times at which a measurement of the object alone has been taken. A blue star indicates that a clutter-generated measurement has been recorded (of course, the algorithm does not know whether the measurement was generated by clutter or by the real object). A black star indicates that both the clutter as well as the object have both been recorded by the sensor. Times for which no stars appear are times during which no measurements were returned either the clutter source as well as the object were out of the field-of-view or the object and clutter source were in view but have not have generated any returns (i.e., misdetections without false alarms). The velocity error is shown in Figure 2

Figure 3 shows the posterior probability $p(t_{k+1})$ that an object exists as a function of time. Figure 4 shows the number of GM components generated by AEGIS. The number of components tended to be large in the initial stages of the simulation, but as more measurements of the object are taken, a fewer number of GM components is generated. Even when the components are larger in number, the simulation takes, typically, less than 1 second per simulation time step (the entire code was written in Matlab and run on a 2.4GHz computer).

Figure 5 shows the hybrid information divergence gained at each time step. We used a FISST version of the gamma information divergence [16] to evaluate the gain. How to compute the gamma divergence in FISST is discussed in [6]. As we can see from the figure, a very large amount of information is gained the first time we detect the object. As long as the object is in the field-of-view, information is gained as tracking measurements are made. Hence, the jump in information divergence is due to the (re)detection of the object (i.e., a gain in information related to the discrete component of the problem) and the subsequent gain in information is due to the tracking of the object's state (i.e., a gain in information related to the continuous component of the problem). Once the object is no longer in the field-of-view, the information gain is zero.

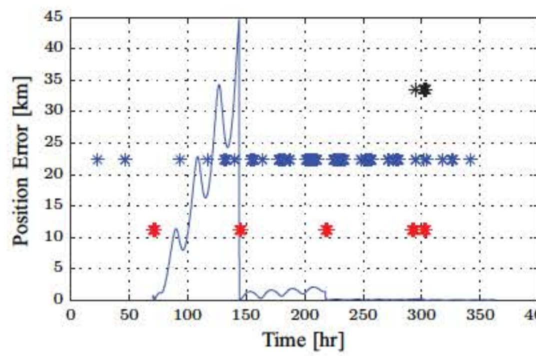

Fig. 1. Magnitude of the true relative position error.

VI. Conclusion

In this paper we used FISST to solve a simple SSA search, detect and track problem. A hybrid estimation technique like

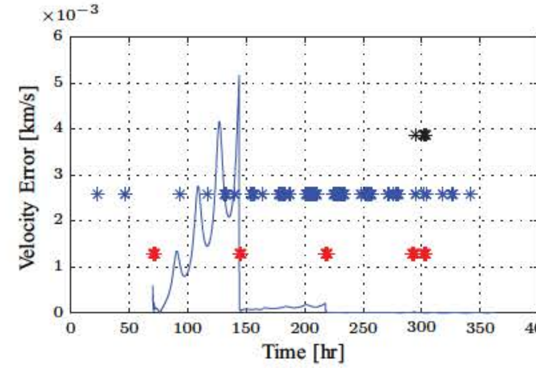

Fig. 2. Magnitude of the true relative velocity error.

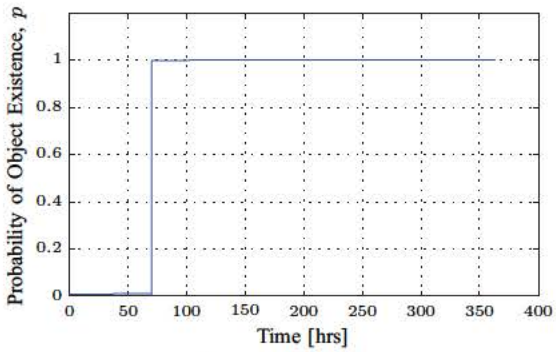

Fig. 3. Probability of object existence.

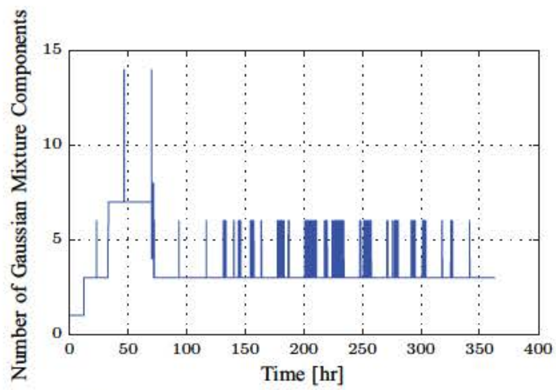

Fig. 4. Number of GM components in AEGIS.

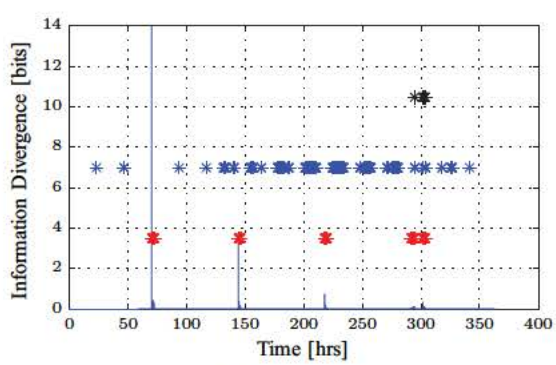

Fig. 5. Information divergence.

FISST is important to a problem like SSA since information loss due to the artificial separation of the detection and tracking problems is eliminated. However, FISST is computationally expensive unless some approximation technique is employed. Unlike the GM-PHD approach, the multi-target pdf is first approximated by its first moment PHD which, in turn, is further approximated using a GM model. In the approach presented in this paper, we do not rely on using the PHD as a first moment approximation of the multi-target pdf. Instead, it is shown that a prior pdf that is represented by a GM implies that the posterior pdf can also be represented by a GM. This should reduce the computational burden by using a GM approximation to FISST, but without employing the first moment PHD approximation.

Future work will focus on extending this solution to a more realistic SSA scenarios with an arbitrary number of objects and clutter sources, both modeled using spatial Poisson distributions. We also plan on applying higher fidelity six degree of freedom dynamics, modeling both three-dimensional translation and full rotational dynamics as well as (conservative and nonconservative) perturbations to the two-body Keplerian motion. Finally, we also seek to include characterization to the approach such that we have a complete integrated detection, characterization and tracking AEGIS-FISST solution to the general SSA problem.

Acknowledgment

The authors wish to acknowledge the financial support of the National Research Council in supporting the research of Prof. Hussein, Dr. DeMars, and Dr. Früh.

References

[1] D. Hall, J. Africano, D. Archambeault, B. Birge, D. Witte, and P. Kervin, "Amos observations of nasa's image satellite," *2006 AMOS Conference*, 2006.

[2] D. Hall, B. Calef, K. Knox, M. Bolden, and P. Kervin, "Separating attitude and shape effects for non-resolved objects," *2007 AMOS Conference*, 2007.

[3] T. Kelecy and M. Jah, "Initial strategies to recover and predict high area-to-mass ratio (hamr) geo space object trajectories with no a priori information," *Journal of the International Academy of Astronautics: Acta Astronautica*, 2010, accepted (05/09/11).

[4] I. R. Goodman, R. P. S. Mahler, and H. T. Nguyen, *Mathematics of Data Fusion*. Kluwer Academic Publishers, 1997.

[5] R. P. S. Mahler, *Statistical Multisource-Multitarget Information Fusion*. Artec House, 2007.

[6] K. J. DeMars, I. I. Hussein, M. K. Jah, and R. S. Erwin, "Information divergence in a gm-fisst approximation," *15th International Conference on Information Fusion*, July 2012.

[7] R. P. S. Mahler, "Multitarget markov motion models," *SPIE Conference on Signal Processing, Sensor Fusion, and Target Recognition*, vol. 3720, pp. 47 56, 1999.

[8] K. J. DeMars, R. H. Bishop, and M. K. Jah, "A splitting gaussian mixture method for the propagation of uncertainty in orbital mechanics," *21st AAS/AIAA Space Flight Mechanics Meeting*, February 2011.

[9] , "Propagation of uncertainty in the presence of attitude-dependent solar radiation pressure effects." *AAS/AIAA Astrodynamics Specialist Conference*, July/August 2011.

[10] G. Terejanu, P. Singla, T. Singh, and P. Scott, "Uncertainty propagation for nonlinear dynamic systems using gaussian mixture models," *Journal of Guidance, Control, and Dynamics*, vol. 31, no. 6, pp. 1623 1633, November December 2008.

[11] K. J. DeMars, "Nonlinear orbit uncertainty prediction and rectification for space situational awareness," Ph.D. dissertation, The University of Texas at Austin, 2010.

[12] R. Vallée, "Information entropy and state observation of a dynamical system," in *Uncertainty in Knowledge-Based Systems*, ser. Lecture Notes in Computer Science, B. Bouchon and R. Yager, Eds. Springer Berlin/Heidelberg, 1987, vol. 286, pp. 403 405.

[13] S. J. Julier, J. K. Uhlmann, and H. F. Durrant-Whyte, "A new approach for filtering nonlinear systems," *Proceedings of the American Control Conference*, vol. 3, pp. 1628 1632, June 1995.

[14] B.-T. Vo, B.-N. Vo, and A. Cantoni, "The cardinality balanced multi-target multi-bernoulli filter and its implementations," *IEEE Transactions on Signal Processing*, vol. 57, no. 2, pp. 409 423, February 2009.

[15] A. E. Roy, *Orbital Motion*. Philadelphia, PA: Institute of Physics Publishing, 1998.

[16] R. Jenssen, "An information theoretic approach to machine learning," Ph.D. dissertation, University of Tromso, 2005.